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A RESOURCE CONFLICT RESOLUTION PROBLEM FORMULATED IN CONTINUOUS TIME

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An unlimited source of messages transmits via a single channel Poisson wise (λ); message holding times are random (Exp(μ)). Conflict and mutual destruction may occur: an attempt to transmit when channel is busy destroys both incumbent and applicant; both revert to a limbo state, reapplying independently according to Exp(ν). Surprisingly, this process may reach statistical equilibrium with a simply solution, which is displayed.

EXECUTIVE SUMMARY

Analysis of procedures for allocating channels to randomly arriving message traffic by mathematical methods aids in making decisions. This paper provides an example. First, we provide some background.

There are situations in which many different message sources independently compete for service at one, or a finite number of channels or other facilities. If one message is on the channel, and another applies, a conflict occurs, and both are destroyed, meaning that they go into a re-try or limbo state, from which they independently attempt to access the channel at random (exponentially distributed) intervals; of course retries can also collide and be destroyed, but the process continues indefinitely.

The usual formulation assumes that messages appear in discrete packets; then, assuming an infinite message source, and that interrupted packets must re-start, the number in limbo will eventually increase indefinitely -- the process is unstable, with delays increasingly to infinity. Only by allowing for takedown or defection in the model can stability be reached.

This report generalizes the classical model slightly to consider a long stream of very small packets. Interruption is still possible, but re-start is not required. It is shown that for this setup stability may be achieved provided demand rate is less than a critical value. The probability distribution of limbo state takes on a simple form, as do the low moments (mean and variance).

A RESOURCE CONFLICT RESOLUTION PROBLEM FORMULATED IN CONTINUOUS TIME

D.P. Gaver
G. Fayolle

1. Introduction

In many situations involving data transmission from diverse sources there can be conflict for a limited number of channels or other facilities. Uncoordinated attempts by several sources to use a single facility can result in "collision," the "destruction" of all participants in the collision, meaning the loss of the transmission, and hence the need for re-transmission. An important problem concerns the development of workable procedures for alleviating the conflict and corresponding message delay problems.

Often such problems are viewed as occurring in discrete time: slots of equal length occur in temporal succession, and each slot can handle just one packet of data at a time, if two or more packets try to use the same slot simultaneously, a collision occurs that somehow must be resolved. A recent paper by Fayolle, Flagolet, and Hofri (1983), hereafter FFH, analyzes a stack protocol for handling such a situation, but there are many other proposals.

This report is concerned with some simple models for contention for a single facility (channel), and for contention or conflict resolution. The models are formulated in a continuous-time manner: messages, or numbers of packets constituting messages, are "long," meaning that they occupy many consecutive slots on the average if a single transmission is occurring.

2. Model 1: Poisson Message Source, Single Facility

Permit messages to arrive at a single facility (e.g., bus or satellite link) in Poisson manner with rate λ. Service times are IIDExp(μ). When a message arrives it either: (i) encounters a free facility and immediately begins transmission, or (ii) interrupts ("collides with," "destroys") a message in progress; the result is that both messages are affected, and some retransmission becomes necessary.

In what follows we investigate a scheme to allow the messages to retransmit following interruption or collision. It can be called a stochastic stacking procedure. In a general sense it is patterned after the algorithm analyzed by FFH.

2.1 Stochastic Stacking Model; A Single Limbo State

Introduce this procedure: whenever a collision occurs, each message source selects a delay time for retransmission independently from the distribution Exp(ν). While any message experiences the latter delay it will be said to occupy a limbo state; the number of messages occupying the limbo state at time t will be denoted by $X(t)$. Furthermore, let $A(t)$ denote the state of the facility at t: $A(t) = 1$ if the facility is occupied with a message transmission, while $A(t) = 0$ if the facility is idle at time t. The idea of the delay time here qualitatively resembles the randomization scheme studied by FFH.

Apparently $\{A(t), X(t);\ 0 \geq t\}$ is a Markov chain in continuous time. The forward Kolmogorov equations for the process can be written in terms of:

$$q_j(t) = P\{X(t) = j, A(t) = 1\} , \tag{2.1a}$$

$$p_j(t) = P\{X(t) = j, A(t) = 0\} , \tag{2.1b}$$

The probabilities in question actually depend upon initial conditions, i.e., values of $X(0)$ and $A(0)$, but these will be left implicit. Consider the evolution of the probabilities as follows:

$$p_j(t+dt) = p_j(t)[1-\lambda dt-j\nu dt]+q_{j-2}(t)\lambda dt+q_{j-1}(t)\nu(j-1)dt$$
$$+ q_j(t)\mu dt + o(dt) \tag{2.2a}$$

$$q_j(t+dt) = q_j(t)[1-\lambda dt-j\nu dt-\mu dt]+p_j(t)\lambda dt$$
$$+ p_{j+1}(t)\nu(j+1)dt + o(dt) . \tag{2.2b}$$

Now subtract $p_j(t)$ $(q_j(t))$ from both sides of (2.2a) (2.2b), divide by dt and let $dt \to 0$ to obtain the formal Kolmogorov equations:

$$\frac{dp_j}{dt} = -(\lambda+\nu j)p_j(t)+\lambda q_{j-2}(t)+\nu(j-1)q_{j-1}(t)+\mu q_j(t) \tag{2.3a}$$

$$\frac{dq_j}{dt} = -(\lambda+\nu j+\mu)q_j(t)+\lambda p_j(t)+\nu(j+1)p_{j+1}(t) \tag{2.3b}$$

A few words of explanation: in order for the system to be in state $(j,0)$ at time $t+dt$ it must have either: (1) been in that state at t, i.e., a moment before, and have experienced no change, <u>or</u> (2) been in state (j-2,1) a moment before, and experienced an exogenous (λ-rate) arrival; the latter collides with the message on the facility, and both enter the limbo state, so the state changes to (j,0), <u>or</u> (3) been in state (j-1,1) a moment before and experienced an endogenous (from limbo state) arrival, <u>or</u> (4) been in state (j,1) and experienced departure of the message on the channel. This explains equation (2.2a); otherwise, in order that the system be in state (j-1,1) at time t+dt it must have either (5) been in that state at t, i.e., a moment before and thus experienced no change, <u>or</u> (6) been in state (j,0) and experienced an exogenous arrival which begins service on the channel, <u>or</u> (7) been in state (j+1,0) a moment before and experienced an endogenous arrival, from a message that leaves the limbo state and entered the idle facility. This explains expression (2.2b).

2.2 <u>Long-Run or Steady State Conditions and Distributions</u>

To look for the conditions allowing a stable long-run distribution of $\{X(t),A(t)\}$ set the time-derivatives to zero in (2.3a and b) and introduce generating functions for the limiting probabilities; by definition

$$P(z) = \lim_{t\to\infty} \sum_{j=0}^{\infty} p_j(t)z^j = \sum_{j=0}^{\infty} p_j z^j , \tag{2.4a}$$

$$Q(z) = \lim_{t\to\infty} \sum_{j=0} p_j(t)z^j = \sum_{j=0} q_j z^j , \tag{2.4b}$$

where the latter limits are assumed to exist at least when $|z| \leq 1$. Performing the summations leads to these equations for the generating functions:

$$\lambda P(z) + \nu z P'(z) = [\lambda z^2+\mu]Q(z) + \nu z^2 Q'(z) \tag{2.5a}$$

$$\lambda P(z) + \nu P'(z) = [\lambda+\mu]Q(z) + \nu z Q'(z) \tag{2.5b}$$

Now multiply (2.5b) through by z, subtract from (2.5a), and divide by (1-z) to obtain

$$\lambda P(z) = (\mu-\lambda z)Q(z) . \tag{2.6}$$

If we put z = 1, this expression results in

$$\lambda[P(1)+Q(1)] = \mu Q(1) ,$$

but since we assume an honest limiting probability exists, i.e., that

$$P(1) + Q(1) = \sum_{j=0}^{\infty} [P\{X=j,A=0\} + P\{X=j,A=1\}] = 1 ,$$

it follows that

$$\sum_{j=0}^{\infty} q_j = Q(1) = \frac{\lambda}{\mu} \equiv \rho \quad (0 \leq \rho < 1) . \tag{2.7a}$$

and that

$$\sum_{j=0}^{\infty} p_j = P(1) = 1-\rho \tag{2.7b}$$

Further information results by differentiating (2.6) and substituting into (2.5b); the result is the simple differential equation

$$\frac{Q'(z)}{Q(z)} = \frac{\lambda(1+z)+\nu}{\nu(\frac{\mu}{\lambda}-2z)} = \frac{\lambda}{\nu}\left[\frac{\lambda(1+z)+\nu}{\mu-2\lambda z}\right]$$

$$= -\frac{\lambda}{2\nu} + \frac{\lambda}{\nu}(\lambda+\nu+\frac{\mu}{2})\frac{1}{\mu-2\lambda z} \tag{2.8}$$

which can easily be integrated to yield

$$\ell n\left[\frac{Q(z)}{Q(1)}\right] = \frac{\lambda}{2\nu}(1-z) + \frac{1}{2\nu}(\lambda+\nu+\frac{\mu}{2})\,\ell n\left(\frac{\mu-2\lambda}{\mu-2\lambda z}\right)$$

or, utilizing the value of $Q(1)$,

$$Q(z) = \rho\left[e^{\frac{\lambda}{2\nu}(1-z)}\left(\frac{1-2\rho}{1-2\rho z}\right)^{\left(\frac{\lambda+\nu+\mu/2}{2\nu}\right)}\right], \tag{2.9}$$

so the ergodicity condition immediately appears to be $0 \leq \rho < \frac{1}{2}$.

It then follows from (2.6) that

$$P(z) = (\frac{1}{\rho}-z)Q(z), \tag{2.10}$$

and hence the generating function of total system occupancy (server plus limbo) is

$$H(z) = E[z^{X+A}]$$

$$= P(z) + zQ(z) = \left[e^{\frac{\lambda}{2\nu}(1-z)} \left(\frac{1-2\rho}{1-2\lambda z}\right)\right]^{\frac{\lambda+\nu+\mu/2}{2\nu}} ; \qquad (2.11)$$

$$0 \leq 2\rho < 1 .$$

Differentiation of $H(z)$ at $z = 1$ generates cumulant-like objects that can be converted to moments. These result in:

$$E[X+A] = \frac{\rho}{1-2\rho}\left(1 + 2\rho\frac{\mu}{\nu}\right) ; \qquad (2.12)$$

$$\mathrm{Var}[X+A] = \frac{\rho\left(1+\rho(3-\rho)\frac{\mu}{\nu}\right)}{(1-2\rho)^2} , \quad 0 \leq 2\rho < 1 \qquad (2.13)$$

Apparently both the mean and variance of total system occupancy decrease monotonically with a decrease in ν^{-1}, the mean delay time selected by (or for) any colliding message. This suggests that the best control policy is to insist that interrupted or destroyed messages should immediately try again for transmission, if one is guided by an assessment of mean system delay time by Little's Formula.

2.3 Inversion of H(z) When Limbo Delays Are Short

Suppose $\nu \to \infty$ in (2.11), signifying retransmission after a negligible delay in the limbo state. Then H(z) approaches

$$H_0(z) = \left(\frac{1-2\rho}{1-2\rho z}\right)^{1/2} ; \quad 0 \leq 2\rho < 1 . \tag{2.14}$$

The latter resembles the ordinary M/M/1 queue generating function, but the power 1/2 instead of 1 is noticeably different. Recognition that the generating function is that of a particular negative binomial distribution yields the explicit formula

$$P\{X+A=j\} = \frac{\Gamma(j+\frac{1}{2})}{j!\ \Gamma(\frac{1}{2})}(1-2\rho)^{1/2}\ (2\rho)^j \ ; \quad j = 0,1,2,... \tag{2.15}$$

Suppose for one moment that perfect information were available, and that arriving messages could be queued before transmission on the system; no collisions can occur. Then the probability distribution of the total number in the system is well-known to be geometric,

$$P\{X+A=j\} = (1-\rho)\ \rho^j , \quad 0 \leq \rho < 1 \tag{2.16}$$

and

$$E[X+A] = \frac{\rho}{1-\rho} \tag{2.17}$$

It then appears from Little's formula that the ratio of long-run expected total delays in the collision-prone but stacked, and the queued systems is at best

$$\frac{E[\text{Delay Stacked}]}{E[\text{Delay Queued}]} = \begin{cases} \dfrac{1-\rho}{1-2\rho}, & 0 \leq \rho < 1/2 \\ 1+\rho+2\rho^2+o(\rho^3) & \text{as} \quad \rho \to 0 \; . \end{cases} \tag{2.18}$$

which clearly reveals the advantage obtainable if information can somehow reduce or remove collision frequency.

3. The Busy Signal Problem.

Consider the following classical problem, historically preceding the conflict problem previously discussed, but of interest in its own right. A telephone line serves a number of customers. At moments of a Poisson process of rate λ customers attempt to initiate calls on the line; if the line is free it is captured by a caller for an exponentially distributed time, mean μ^{-1}. If a call is in progress when another call arrives the newcomer hears a busy signal, hangs up and tries again (retries) after an exponentially distributed time, mean ν^{-1}; he continues to re-try, along with others who have experienced busy signals, in such a manner, i.e. at independent exponential (ν) intervals until he accesses the line and can initiate, and eventually complete, his call.

The above familiar setup is very similar to the conflict resolution problem just addressed, but has no "collision" or "destruction" features. Note, however, that certain proprietary telephone systems do have a destruction feature for low priority calls. Thus the U.S. Dept. of Defense AUTOVON system allows high priority calls to displace ones of low priority. It seems reasonable to model this latter situation using a limbo or retry state much as we did the previous conflict problem. The present discussion models only the single-priority setup.

3.1 Probabilities for A Single Limbo or Retry State.

Again consider the vector Markov chain $\{A(t),X(t),t \geq 0\}$, where $A(t) = 1$ if the line is occupied, and $= 0$ if the line is free, while $X(t)$ is the number in the limbo/retry state. Again

$$q_j(t) = P\{X(t)=j, A(t)=1\} \qquad (3.1,a)$$

$$p_j(t) = P\{X(t)=j, A(t)=0\} \; ; \qquad (3.1,b)$$

we will write formal Kolmogorov equations to describe the evolution of the probabilities:

$$p_j(t+dt) = p_j(t)[1-\nu j dt-\lambda dt] + q_j(t)\mu dt + o(dt) \qquad (3.2,a)$$

$$q_j(t+dt) = q_j(t)[1-\nu j dt-\lambda dt-\mu dt] + q_{j-1}(t)\lambda dt$$

$$+ p_{j+1}(t)\nu(j+1)dt + p_j(t)\lambda dt + q_j(t)\nu j dt \; . \qquad (3.2,b)$$

The usual steps yield the differential equations

$$\frac{dp_j}{dt} = -(\lambda + \nu j)p_j(t) = \mu q_j(t) \qquad (3.3,a)$$

$$\frac{dq_j}{dt} = -(\lambda+\mu)q_j(t) + \lambda q_{j-1}(t) + \nu(j+1)p_{j+1}(t) + \lambda p_j(t) \; . \qquad (3.3,b)$$

Notice that the effect of one possible change, i.e. that in which a retry population number retries and again gets a busy signal, can be removed, for there is no net change in state.

3.2 The Long-Run Distribution.

Assume now that a long-run distribution occurs, and search for the necessary conditions and the distributional form. We must solve the balance equations obtained by zeroing the derivatives in (3.3):

$$(\lambda+\nu j)p_j = \mu q_j \tag{3.4,a}$$

$$(\lambda+\mu)q_j = \lambda q_{j-1} + \nu(j+1)p_{j+1} + \lambda p_j \ . \tag{3.4,b}$$

Introduce generating functions

$$P(z) = \sum_{j=0}^{\infty} z^j p_j \ , \quad Q(z) = \sum_{j=0}^{\infty} z^j q_j \ . \tag{3.5}$$

With a little calculation it can be shown that

$$\lambda P(z) + \nu z P'(z) = \mu Q(z) \tag{3.6,a}$$

$$(\lambda+\mu)Q(z) = \lambda z Q(z) + \nu P'(z) + \lambda P(z) \tag{3.6,b}$$

by multiplying (3.4,a) and (3.4,b) through by z^j and summing, as before; the primes denote z-differentiation. To solve, multiply (3.6,b) by z and subtract from (3.6,a) to obtain

$$\lambda[P(z) + zQ(z)] = \mu Q(z) \tag{3.7}$$

after division by 1-z; whence

$$\lambda[P(1) + Q(1)] \equiv \lambda = \mu Q(1)$$

so

$$\lim_{t\to\infty} P\{A(t) = 1\} = Q(1) = \frac{\lambda}{\mu} \equiv \rho , \tag{3.8}$$

the probability of a busy line in the long run ($\rho < 1$). Next rewrite (3.7) as

$$P(z) = (1/\rho - z)Q(z) \tag{3.9}$$

and differentiate,

$$P'(z) = (1/\rho - z)Q'(z) - Q(z) ; \tag{3.10}$$

finally substitute for P and P' from (3.9) and (3.10) into (3.6,a) to obtain the differential equation

$$Q'(z) = \frac{1 + \lambda/\nu}{1/\rho - z} Q(z) \tag{3.11}$$

which is immediately solved to yield

$$Q(z) = \rho\left(\frac{1 - \rho}{1 - \rho z}\right)^{1+\lambda/\nu} \tag{3.12}$$

an expression for P(z) comes from (3.9), and finally

$$\lim_{t\to\infty} E[z^{X(t)+A(t)}] = P(z) + zQ(z) = \left(\frac{1 - \rho}{1 - \rho z}\right)^{1+\lambda/\nu}. \tag{3.13}$$

It follows by inspection that the stationary distribution of system occupancy, including the channel occupant and retry population, is now negative binomial:

$$\lim_{t\to\infty} P\{X(t)+A(t)=j\} = \frac{\Gamma(1+\lambda/\nu+j)}{j!\,\Gamma(1+\lambda/\nu)}\,\rho^j(1-\rho)^{1+\lambda/\nu}\,. \qquad (3.14)$$

Note that if $\nu \to \infty$ the generating function of (3.13) tends to $(1-\rho)/(1-\rho z)$, that of the geometric distributionof occupancy of the M/M/1 system that permits queueing. This is quite intuitive, for infinitely frequent retries look to the system--if not the customer--exactly as if arrivals are queued. There is a decided difference between (3.13), or even (3.13) with $\nu \to \infty$, and the corresponding collision-destruction model, with generating function (2.11) or (2.1).

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SOME COMMENTS ON A "RESOURCE CONFLICT RESOLUTION PROBLEM FORMULATED IN CONTINUOUS TIME"

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It is well known that slotted Aloha schemes are strongly unstable when there are an infinite number of sources. Yet Gaver and Fayolle [G-F], in the paper referred to in the title, find that, for $2\rho < 1$, a continuous time Aloha scheme is stable. In this note we try to explain this discrepancy via some elementary calculations which illuminate the structure of the model analyzed in [G-F]. We use the notation of [G-F].

Consider $X(t) + A(t)$, the total number of messages in the system at time t. We shall see that this process is approximately a birth-death process with birth rate λ and death rate $\mu/2$ when $X(t) + A(t)$ is large. Furthermore, we will show that $A(t) = 1$ about half the time when $X + A$ is large, and we will develop some further consequences of our viewpoint.

To begin with, suppose that $X(0) = K$ is large and that $A(0) = 0$. After a time which is distributed exponentially with mean $\dfrac{1}{\lambda + K\nu} \approx \dfrac{1}{K\nu}$, either a message goes from limbo to transmission (with probability $\dfrac{K\nu}{\lambda + K\nu} \approx 1 - \dfrac{\lambda}{K\nu}$) or a new message arrives (probability $\dfrac{\lambda}{\lambda + K\nu} \approx \dfrac{\lambda}{K\nu}$). Obviously, the most likely event is that a message goes from limbo to transmission. In that case, after another exponentially distributed time with mean $\dfrac{1}{\lambda + (K-1)\nu + \mu} \approx \dfrac{1}{K\nu}$, either another message from limbo collides with the first (probability $\dfrac{(K-1)\nu}{\lambda + (K-1)\nu + \mu} \approx 1 - \dfrac{\mu + \lambda}{K\nu}$), the first message completes transmission (probability $\dfrac{\mu}{\lambda + (K-1)\nu + \mu} \approx \dfrac{\mu}{K\nu}$), or a new message arrives and collides with the first (probability $\dfrac{\lambda}{\lambda + (K-1)\nu + \mu} \approx \dfrac{\lambda}{K\nu}$). In any case we see that in a (short) time

$$\Delta t \approx \frac{2}{K\nu}$$

we have a new arrival with probability

$$p\ (\text{new}) \approx \frac{2\lambda}{K\nu} \approx \lambda\Delta t\ ,$$

a successful transmission with probability

$$p\ (\text{success}) \approx \frac{\mu}{K\nu} \approx \frac{\mu}{2}\Delta t\ ;$$

and a return to the initial condition $X = K$, $A = 0$ with probability

$$p\ (\text{return}) \approx 1 - \lambda\Delta t - \frac{\mu}{2}\Delta t\ .$$

That is, the process is approximately birth-death with birth rate λ and death rate $\mu/2$. Furthermore, the channel is idle for times of length $\approx \frac{1}{K\nu}$ and is busy for times of length $\approx \frac{1}{K\nu}$, so that half the time is spent busy (this is another way of saying that the death rate is $\frac{\mu}{2}$). This clearly shows that the stability condition is $\frac{2\lambda}{\mu} = 2\rho < 1$. While the preceding argument is not rigorous, it is not hard to tighten up, but in light of [G-F] there seems to be no point in doing so.

Let us carry our reasoning a bit further. We have $(X(t) + A(t))$ is approximately a constant-coefficient birth-death process for large values of $X + A$; hence in steady state, we should have as K becomes large,

$$P(X(t) + A(t) = K) \approx C\,(2\rho)^K$$

for some $C > 0$. Using the notation in [G-F] this is

$$p_K + q_{K-1} \approx C\,(2\rho)^K\ . \tag{1}$$

Also, since the process spends about the same amount of time in state $(X = K, A = 0)$ as in $(X = K-1,\ A = 1)$, we should have

$$p_K \approx q_{K-1}\ . \tag{2}$$

Together these two equations imply that

$$\lambda p_K = \mu q_K - \lambda q_{K-1}\ ,$$

which is the key equation (2.6) in [G-F]. Furthermore, equation (1) is the same as their equation 2.16 if we replace C by $\frac{2}{\sqrt{\Pi}}\sqrt{1-\rho}$.

Now our reasoning was valid as long as $K\nu$ was large. Instead of assuming that K was large, we could as well have supposed that ν was large. In this case we have

$$\Delta t \approx \frac{1}{K\nu} + \frac{1}{(K-1)\nu} = \frac{2K-1}{K(F-1)\nu}$$

$$p\ (\text{successes}) \approx \frac{\mu}{(K-1)\nu} = \frac{\mu}{2-\frac{1}{K}} \Delta t\ .$$

The birth rate, of course, remains λ. Then by a basic formula of birth-death processes

$$P(\text{state} = K) = C \prod_{j=1}^{K} \frac{\lambda_j}{\mu_j} \quad \text{for some } C > 0$$

we have

$$P(X(t) + A(t) = K) = p_K + q_{K-1}$$

$$= C \prod_{j=1}^{K} \frac{\lambda}{\mu} \left[2 - \frac{1}{j}\right] = C\rho^K \prod_{j=1}^{K} \frac{2j-1}{j}$$

$$= C\rho^K \frac{2^K \Gamma\left(K + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = C \frac{\Gamma\left(K+\frac{1}{2}\right)}{\sqrt{\Pi}\, K!} (2\rho)^K$$

which is the equation (2.15) in [G-F] for the choice $C = \sqrt{1-2\rho}$.

We can also analyze the effects of changing the model. In [G-F], any transmission, no matter how short, reduces the remaining length of a message. In fact, we saw that when $X(t)$ is large, the system takes very small "bites" very quickly. Now a realistic system may have some overhead, say a time δ, which must pass before a new transmission may start. Or there may be some basic unit to be transmitted such as a packet or length δ, and if the transmission is interrupted in the middle of a packet then the entire packet must be retransmitted. Or there may be a time delay δ through the system so that when an interruption takes place, the last δ of the message must be retransmitted to assure its safe delivery. We model any of these cases by assuming that transmissions which are shorter than a fixed time δ are ineffective.

We shall now demonstrate that this system is strongly unstable; that is

$$P(\lim_{t \to \infty} X(t) = \infty) = 1.$$

Furthermore we shall show that

$$P \text{ (total number of successful transmissions is finite)} = 1;$$

that is, we will show that, with probability 1, there is a random finite time T such that no successful transmissions take place after time T. To this end, consider the probability that at least one message which attempts to transmit during a time when $X(t) + A(t) = K$ finds no uninterrupted interval of length at least δ. This is bounded by the probability that in a time interval of exponentially distributed (λ) length, a Poisson process with rate $K\nu$ has no consecutive events more than δ apart; that is,

$$P \text{ (no transmission)} \geq \sum_{j=1}^{\infty} (1 - e^{-\delta K\nu})^{j+1} \left(1 - \frac{\lambda}{K\nu + \lambda}\right)^{j} \frac{\lambda}{K\nu + \lambda}$$

$$= \frac{\lambda}{K\nu + \lambda} (1 - e^{-\delta K\nu}) \frac{1}{1 - (1 - e^{-\delta K\nu})(1 - \frac{\lambda}{K\nu + \lambda})}$$

$$= 1 - K\nu e^{-\delta K\nu} + 0(e^{-\delta K\nu})$$

That is,

$$P(\text{transmission}) \leq K\nu e^{-\delta K\nu} + 0(e^{-\delta K\nu})$$

Hence by a Borel-Cantelli argument we may conclude that

$$P \text{ (infinite number of transmissions)} = 0$$

Since

$$\sum_{K=N}^{\infty} K\nu e^{-\delta K\nu} + 0(e^{-\delta K\nu}) < \infty$$

for all positive values of N, K and δ.

Suppose now that the overhead δ is a random variable instead of a constant. Then if $P(\delta = 0) = 0$, the queue is strongly unstable, although for some distributions of δ an infinite number of messages may get through. This can be seen as follows: work is done on a message only after overhead is completely serviced. If the overhead distribution has a continuous density $f(X)$ near $X = 0$, then successful completion of overhead occurs approximately as a Poisson process with rate $f(0)/2$. After each successful overhead completion, a message gets about $1/\nu X(t)$ time to transmit. Now $X(t)$ will clearly be on the order of λt for large t, so successful message transmissions will occur at a rate of about $f(0)/2\lambda t \nu$ for large t. The analysis of other types of overhead distributions is straightforward and is not defined here.

Let us now suppose that message lengths are not exponentially distributed, but have instead a distribution function $F(X)$. Then it is not hard to see that if the failure rate of $F(X)$ (namely $\frac{F'(X)}{1-F(X)}$) remains above a level μ where $2\lambda/\mu < 1$, then the system is stable. This follows directly from our original analysis. As our final model, consider the differences between Gaver calls "resume," "restart," and "new" disciplines. "Resume" is the original model, where each transmission, no matter how short, reduces the remaining length of a message. "Restart" is an all-or-nothing situation. Either the entire message gets transmitted, or if interrupted it must be entirely redone. "New" is where each message picks a fresh length at the start of each transmission, independently from a distribution $F(X)$. That is, each message length is a new i.i.d. random variable in each transmission attempt. This might arise, for instance, in a simulation where the programmer does not keep track of each message length separately, but instead picks a new length each time. The first case (resume) has already been discussed. "Restart" is easily seen to be strongly unstable by the same sort of reasoning we used for analyzing overhead, and the reader is invited to fill in the details. We shall now show that when $F(X)$ has a continuous density $f(X)$ near 0, "new" is stable if and only if $2\lambda/f(0) < 1$. That is, the three cases are quite different, and so simulations must be done quite carefully. The analysis is nearly identical to the "resume" case: in each time interval of length $\Delta t \approx 2/K\nu$, we have a probability of abut $f(0)\Delta t/2$ of a successful transmission. In fact the probability of a successful transmission is half the probability that a random variable with distribution function $F(X)$ is less than an independent exponentially distributed variable with mean $1/K\nu$, so

$$P(\text{successes}) \approx \frac{1}{2} \int_0^\infty e^{-K\nu s} f(s)ds$$

$$\approx f(0)/2K\nu \,.$$

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